Many Faces of Logic

Dirk Hofmann and Manuel A. Martins*

University of Aveiro, Department of Mathematics, Aveiro, Portugal
<{dirk,martins}@ua.pt>

Abstract. In this paper we present logic from various perspectives, starting from the standard way typically taught in an undergraduate course. We expose the relationship with other mathematical structures, namely closure relations, closure operators, coalgebras and bialgebras.

Introduction

No doubt, there are several ways of presenting logic. Here we focus on two particular viewpoints which emphasises different aspects: algebraic logic and abstract logic. The former is more intuitive and therefore more often used in class room, while the latter considers only the formal properties of deduction ignoring the structure of formulae.

The algebraic approach to sentential logic is very powerful, since it allows the use of tools and results from universal algebra (e.g. ultraproducts) to study logical systems. One important methodology is the classical Lindenbaum-Tarski process which associates to a sentential logic a class of algebras. Paradigmatic examples are the Boolean algebras in classical propositional logic and Heyting algebras for the intuitionistic propositional logic. These classes of algebras can be viewed as the algebraic counterpart of its corresponding logic in the sense that there is a close relationship between the deductive theory of the logic and the equational theory of the algebras. Abstract algebraic logic goes further, the focus is no longer on the algebraic form of specific logical systems, but on the process of algebraisation itself (cf. [1]).

In the second part of this paper we present logic (to be more precise: a consequence relation) on an abstract set rather then a set of formulae. We exhibit various ways to encode this structure which have roots in different fields of mathematics, namely topology and (co)algebra. We emphasise that properties of these structures and of maps between them can be expressed by simply (in)equalities and with the help of suitable defined compositions, which is useful when transporting notions or ideas from one structure to the other since the transition maps preserve both composition and inequalities. Our presentation here rests partially on general results of [8].

1 Algebraic logic

A *propositional language type* is any set Λ. The elements of Λ are called *functional symbols* in the algebraic context or *logical connectives* in the logical one. With Λ we associate an *arity* function $\rho : \Lambda \to \omega$ such that $\rho(f)$ is the *arity* of the connective $f \in \Lambda$. For each $n \in \omega$, $\Lambda_n := \{f \in \Lambda \mid \rho(f) = n\}$. An *algebra* $\mathbf{A}$ of type Λ is a pair $\langle A, \Lambda^{\mathbf{A}}\rangle$, where A is a non-empty set called the *universe* of $\mathbf{A}$ and $\Lambda^{\mathbf{A}} = \langle f^{\mathbf{A}} \mid f \in \Lambda\rangle$ is a list of operations over

* Partial financial assistance by Centro de Investigação e Desenvolvimento em Matemática e Aplicações da Universidade de Aveiro/FCT and the project MONDRIAN (under the contract PTDC/EIA-CCO/108302/2008) is gratefully acknowledged. The second author also acknowledges partial financial assistance by the project “Nociones de Completud”, reference FFI2009-09345 (MICINN - Spain).

the set A such that, for every $f \in \Lambda_n$, $f^{\mathbf{A}} : A^n \to A$. If $\mathbf{A}$, $\mathbf{B}$ are algebras of the same type, then a mapping $h : A \to B$ is called a *homomorphism of* $\mathbf{A}$ *into* $\mathbf{B}$ (written $h : \mathbf{A} \to \mathbf{B}$), if for every $f \in \Lambda_n$ and every $a_1, \ldots, a_n \in A$, $h(f^{\mathbf{A}}(a_1, \ldots, a_n)) = f^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$. A homomorphism $h : \mathbf{A} \to \mathbf{A}$ is called an *endomorphism of* $\mathbf{A}$.

Let Var be a countable infinite set of *propositional variables*. The set $\mathrm{Fm}_\Lambda\,\mathrm{Var}$ of *formulae of type* Λ *over the set of variables* Var is defined recursively as follows

1. $\mathrm{Var} \subseteq \mathrm{Fm}_\Lambda\,\mathrm{Var}$,
2. if $f \in \Lambda_n$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Fm}_\Lambda\,\mathrm{Var}$, then $f(\alpha_1, \ldots, \alpha_n) \in \mathrm{Fm}_\Lambda\,\mathrm{Var}$.

We can introduce the structure of an algebra on $\mathrm{Fm}_\Lambda\,\mathrm{Var}$ by associating with each $f \in \Lambda_n$ an n-ary operation $f^{\mathbf{Fm}_\Lambda\,\mathrm{Var}}$ on the set $\mathrm{Fm}_\Lambda\,\mathrm{Var}$ defined by $f^{\mathbf{Fm}_\Lambda\,\mathrm{Var}}(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n)$. The algebra $\mathbf{Fm}_\Lambda\,\mathrm{Var}$ is in fact an *absolutely free algebra over the set* X in the class of all algebras of type Λ. An endomorphism $h : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$ is called a *substitution*.

The algebraic approach to sentential logic is very worthwhile. For example, besides the possibility of using tools and results from universal algebra, it also provides a way to establish semantics for logics by considering pairs, algebra together with a set of truth values, as models (usually called matrices). The well known algebraic completeness theorems can be seen as special cases of a general result built on special matrix semantics (cf. [9]).

Traditional notion of deduction. In first contact, logic is usually presented by axioms and inference rules in the so called Hilbert style. For example, the classical propositional logic for the propositional language type $\mathcal{L} = \{\neg, \to\}$ is introduced by the set of axioms $\{p \to (q \to p), (p \to (q \to r)) \to ((p \to q) \to (q \to r)), (\neg q \to \neg p) \to (p \to q)\}$ together with the *Modus Ponens* inference rule $\dfrac{p \to q,\, p}{q}$.

The concept of deduction is defined as follows. An *inference rule* is a pair $\langle \Gamma, \varphi \rangle$ $\Big($also written as $\dfrac{\Gamma}{\varphi}\Big)$ where Γ is a finite set of formulae (the *premises* of the rule) and φ is a single formula (the *conclusion* of the rule). An *axiom* is an inference rule with $\Gamma = \emptyset$, i.e., a pair $\langle \emptyset, \varphi \rangle$, usually just denoted by φ. The rules of this type are called *Hilbert-style* rules of inference.

Let AX be a set of axioms and IR a set of inference rules. We say that a formula φ is *directly derivable* from a set Γ of formulae by the inference rule $\langle \Delta, \psi \rangle$ if there is a substitution h such that $h(\psi) = \varphi$ and $h[\Delta] \subseteq \Gamma$.

We say that ψ is *derivable* from Γ by the set AX and the set IR, in symbols $\Gamma \vdash_{\mathrm{AX,IR}} \psi$, if there is a finite sequence of formulae, $\psi_0, \ldots, \psi_{n-1}$ such that $\psi_{n-1} = \psi$, and for each $i < n$ one of the following conditions hold:

1. $\psi_i \in \Gamma$,
2. ψ_i is a substitution instance of a formula in AX
3. ψ_i is directly derivable from $\{\psi_j : j < i\}$ by one of the inference rules in IR.

Finally, a *proof* is just a derivation from $\emptyset$. The last formula of a proof is called a *theorem*.

Logic as a consequence relation. A *logic* S (or *logical system*) over a propositional language type $\mathcal{L}$ is defined as a pair $S = \langle \mathcal{L}, \vdash_S \rangle$, where $\vdash_S$ is a relation between set of formulae and individual formulae, called the *consequence relation* of S, which satisfies the following conditions, for all $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$:

1. $\varphi \in \Gamma \Rightarrow \Gamma \vdash_S \varphi$ — Reflexivity
2. $\Gamma \vdash_S \varphi$ and $\Gamma \subseteq \Delta \Rightarrow \Delta \vdash_S \varphi$ — Cut
3. $\Gamma \vdash_S \varphi$ and $\Delta \vdash_S \psi$ for every $\psi \in \Gamma \Rightarrow \Delta \vdash_S \varphi$ — Weakening
4. $\Gamma \vdash_S \varphi \Rightarrow h[\Gamma] \vdash_S h(\varphi)$ for every substitution h — Structurality

where $\Gamma \vdash_S \varphi$ abbreviates that $\langle \Gamma, \varphi \rangle \in S$ and reads Γ entails φ in S or φ is a consequence of Γ in S. Note that the reflexivity and weakening conditions together imply the cut condition. We say that $\vdash_S$ is *finitary* if $\Gamma \vdash_S \varphi$ implies $\Gamma' \vdash_S \varphi$ for some finite $\Gamma' \subseteq \Gamma$. The relation $\vdash_{\mathrm{AX,IR}}$ clearly satisfies reflexivity, cut, weakening, structurality and finitary conditions, hence $\langle \mathcal{L}, \vdash_{\mathrm{AX,IR}} \rangle$ is a finitary logic, called the *deductive system* with the set of axioms AX and the set of inference rules IR.

In general, a pair $\langle \mathrm{AX}, \mathrm{IR} \rangle$ of axioms and inference rules such that $\vdash_S = \vdash_{\mathrm{AX,IR}}$ is called an *axiomatization* of S. If both the set of axioms and the set of inference rules are finite then $\langle \mathrm{AX}, \mathrm{IR} \rangle$ is called a *finite axiomatization*. Of course, a deductive system may have several axiomatizations. For instance, given a finitary logic $S = \langle \mathcal{L}, \vdash_S \rangle$, then $\mathrm{AX} := \{\varphi : \emptyset \vdash_S \varphi\}$ and $\mathrm{IR} := \{\langle \Gamma, \varphi \rangle : \Gamma \vdash_S \varphi \text{ and } \Gamma \text{ is finite}\}$ is an axiomatization of S.

A formula is called a *theorem* of S if $\emptyset \vdash_S \varphi$. The set of all theorems is denoted by $\mathrm{Thm}(S)$. A set T of formulae is called a *theory* of S if it is closed under the consequence relation $\vdash_S$, that is, if, for every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$, $T \vdash_S \varphi$ implies $\varphi \in T$. The set of all theories of S is denoted by $\mathrm{Th}(S)$. The set $\mathrm{Th}(S)$ forms a complete lattice $\mathbf{Th}(S) = \langle \mathrm{Th}(S), \cap, \vee^S \rangle$, where the meet operation is the intersection of an arbitrary family of theories and the join operation is defined in the following way: for any $T, T' \in \mathrm{Th}(S)$, $T \vee^S T' = \bigcap \{R \in \mathrm{Th}(S) : T \cup T' \subseteq R\}$. The largest theory is the set $\mathrm{Fm}_{\mathcal{L}}$ and the smallest theory is the set $\mathrm{Thm}(S)$. For any $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$, we denote by $\mathrm{Cn}_S \Gamma$ the smallest theory for S including Γ, i.e., $\mathrm{Cn}_S \Gamma = \{\varphi \in \mathrm{Fm}_{\mathcal{L}} : \Gamma \vdash_S \varphi\}$ and we said that Γ *generates* $\mathrm{Cn}_S \Gamma$. It is not difficult to see that $T \vee^S T' = \mathrm{Cn}_S(T \cup T')$, i.e., $T \vee^S T'$ is the theory generated by $T \cup T'$. A theory T of S is *finitely axiomatized* if $T = \mathrm{Cn}_S \Gamma$ for some finite $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$.

2 Logic, abstractly

In the previous section we have presented the classical view on (propositional) logic. Typically, one starts with a collection of connectives (=operation symbols) which, together with a set of variables, is used to specify what counts as a formula. Furthermore, a consequence relation on the set of formulae can be defined in various ways, here we presented a Hilbert-style calculus with axioms and inference rules. Other possibilities include Gentzen-style calculus, natural deduction, or even a semantical definition. In any case, eventually one arrives at a logical system as described in the previous section. It turns out that such kind of relations on a set X (of formulae) actually appear in many fields of mathematics, and can be described by formally different but equivalent mathematical structures. In this section we shall exhibit a few of them and explain the relationship between them.

A relational view. A *consequence relation* $\vdash$ on a set X is a relation $\vdash : PX \relbar\joinrel\mapstochar\joinrel\rightarrow X$ between subsets of X and points of X which satisfies

1. if $x \in A$, then $A \vdash x$,
2. if $A \vdash x$ and $A \subseteq B$, then $B \vdash x$, and
3. if $A \vdash y$ for all $y \in B$ and $B \vdash x$, then $A \vdash x$;

for all $A, B \subseteq X$. In other words, we require the reflexivity, weakening and cut rule but cannot anymore insist on structurality simply because our "formulae" are now structureless points of an abstract set. Thanks to the second condition above, one can substitute the first one by

1'. $\{x\} \, a \, x$ for all $x \in X$.

The pair $(X, \vdash)$ one calls an *abstract logic*. Given also a set Y with a consequence relation $\Vdash$ and a map $f : X \to Y$, one says that f is *consequence preserving* whenever $A \vdash x$ implies

$f(A) \Vdash f(x)$, for all $A \subseteq X$ and $x \in X$, and f is called *conservative* if $A \vdash x \iff f(A) \Vdash f(x)$.

The axioms defining a consequence relation can be elegantly expressed using the calculus of relations as we explain next. Firstly, recall that for relations $r : X \relbar\joinrel\mapsto Y$ and $s : Y \relbar\joinrel\mapsto Z$, one calculates the composite relation $s \cdot r : X \relbar\joinrel\mapsto Z$ as $x\,(s \cdot r)\,z \iff \exists y \in Y\,(x\,r\,y)\,\&\,(y\,s\,z)$. Secondly, note that every relation $r : X \relbar\joinrel\mapsto Y$ can be lifted to a relation $\hat{P}r : PX \relbar\joinrel\mapsto PY$ between the powersets of X and Y via

$$A\,(\hat{P}r)\,B \text{ whenever } \forall y \in B\,\exists x \in A\,.\,x\,r\,y,$$

for all $A \subseteq X$ and $B \subseteq Y$. We can rewrite now the axioms of a consequence relation as simple *reflexivity* and *transitivity* conditions:

$$\{x\} \vdash x \qquad \text{and} \qquad (\mathcal{A}\,(\hat{P}\vdash)\,A\ \&\ A\,a\,x) \Rightarrow (\bigcup \mathcal{A})\,a\,x,$$

for all $\mathcal{A} \in PPX$, $A \in PX$ and $x \in X$. Equivalently, and without referring to points, these conditions read as

$$1_X \subseteq (\vdash \cdot e_X) \qquad \text{and} \qquad (\vdash \cdot \hat{P}\vdash) \subseteq (\vdash \cdot m_X),$$

where $e_X : X \to PX$, $x \mapsto \{x\}$ and $m_X : PPX \to PX$, $\mathcal{A} \mapsto \bigcup \mathcal{A}$. Finally, when writing $e_X^\circ : PX \relbar\joinrel\mapsto X$ and $m_X^\circ : PX \relbar\joinrel\mapsto PPX$ for the inverse image relations of the functions e_X and m_X, these conditions become

$$e_X^\circ \subseteq \vdash \qquad \text{and} \qquad (\vdash \cdot (\hat{P}\vdash) \cdot m_X^\circ) \subseteq \vdash .$$

In general, for relations $r : PX \relbar\joinrel\mapsto Y$ and $s : PY \to Z$, we can think of $s \circ r := s \cdot (\hat{P}r) \cdot m_X^\circ$ as a kind of composite relation $s \circ r : PX \relbar\joinrel\mapsto Z$. Furthermore, this composition is associative and has the relations $\Delta_X : PX \relbar\joinrel\mapsto X$ defined by $A\,\Delta_X\,x \iff x \in A$ as "weak" identities since

$$r \subseteq \Delta_Y \circ r \qquad \text{and} \qquad r \subseteq r \circ \Delta_X,$$

and one has even equality if and only if $r : PX \relbar\joinrel\mapsto Y$ is *monotone*, that is, $A\,r\,y$ and $A \subseteq B$ imply $B\,r\,y$. Since every consequence relation must satisfy $\Delta_X \subseteq \vdash$, we can think of an abstract logic as a *monoid* $\vdash$ with unit $\Delta_X \subseteq \vdash$ and multiplication $(\vdash \circ \vdash) \subseteq \vdash$.

To every function $f : X \to Y$ we associate monotone relations

$$f_\# : PX \relbar\joinrel\mapsto Y,\ A\,f_\#\,y \iff y \in f(A)$$

and, similarly, $f^\# : PY \relbar\joinrel\mapsto X$ where $B\,f^\#\,x \iff x \in f^{-1}(B)$. For relations $r : PZ \relbar\joinrel\mapsto Y$ and $s : PY \to Z$ one has

$$f^\# \circ r = f^{-1} \cdot r \qquad \text{and} \qquad s \circ f_\# = s \cdot Pf,$$

and therefore also

$$\Delta_X \subseteq f^\# \circ f_\# \qquad \text{and} \qquad f_\# \circ f^\# \subseteq \Delta_Y.$$

The latter inequalities tell us that $f_\#$ and $f^\#$ form an adjunction $f_\# \dashv f^\#$. Furthermore, a map $f : X \to Y$ between abstract logics $(X, \vdash)$ and $(Y, \Vdash)$ is consequence preserving if and only if

$$(f_\# \circ \vdash) \subseteq (\Vdash \circ f_\#).$$

This in turn is equivalent to

$$\vdash \subseteq (f^\# \circ \Vdash \circ f_\#),$$

which indeed reduces to

$$\vdash \subseteq (f^{-1} \cdot \Vdash \cdot Pf),$$

or, equivalently $(f \cdot \vdash) \subseteq (\Vdash \cdot Pf)$.

A relation $r : X \relbar\joinrel\mapsto Y$ is essentially the same thing as a function $\ulcorner r \urcorner : Y \to PX$, via $\ulcorner r \urcorner(y) = \{x \in X \mid x\, r\, y\}$ and $x\, r\, y \iff x \in \ulcorner r \urcorner(y)$. Therefore a relation $r : PX \relbar\joinrel\mapsto Y$ corresponds to both a mapping

$$\mathcal{C}(r) : PX \to PY,\ A \mapsto \{y \in Y \mid A\, r\, y\}$$

and a mapping

$$\mathcal{U}(r) : Y \to PPX,\ y \mapsto \{A \subseteq X \mid A\, r\, y\}.$$

Furthermore, $r : PX \relbar\joinrel\mapsto Y$ is monotone if and only if the map $\mathcal{C}(r) : PX \to PY$ is monotone, if and only if the function $\mathcal{U}(r) : Y \to PPX$ takes value in the set $UX = \{\mathcal{A} \subseteq PX \mid \mathcal{A} \text{ is up-closed}\}$. Here we call a subset $\mathcal{A} \subseteq PX$ *up-closed* if $A \in \mathcal{A}$ and $A \subseteq B$ imply $B \in \mathcal{A}$. In the next two sections we will explore both point of views.

A topological view. In the last section we have seen that every monotone relation $r : PX \relbar\joinrel\mapsto Y$ corresponds precisely to a monotone mapping $\mathcal{C}(r) : PX \to PY$, moreover, one easily verifies that $\mathcal{C}$ preserves composition in the sense that

$$\mathcal{C}(\Delta_X) = 1_{PX} \qquad \text{and} \qquad \mathcal{C}(s \circ r) = \mathcal{C}(s) \cdot \mathcal{C}(r),$$

and that $\mathcal{C}(r) \leq \mathcal{C}(r)$ whenever $r \subseteq r'$. From this it follows at once that consequence relations $\vdash$ on a set X correspond precisely to monotone maps $c := \mathcal{C}(\vdash) : PX \to PX$ satisfying $1_{PX} \leq c$ and $c \cdot c \leq c$, that is,

1. $A \subseteq B \Rightarrow c(A) \subseteq c(B)$,
2. $A \subseteq c(A)$,
3. $c(c(A)) \subseteq c(A)$;

for all $A, B \subseteq X$. Note that one actually has equality in (3), thanks to (2). In generally, a function $c : PX \to PX$ satisfying the conditions above is called a *closure operator*, and the pair (X, c) one calls a closure space.

A map $f : X \to Y$ between closure spaces (X, c) and (Y, d) is called *continuous* whenever f preserves closure points in the sense that $f(c(A)) \subseteq d(f(A))$, for all $A \subseteq X$. This can be equivalently expressed in the calculus of relations as $Pf \cdot c \leq d \cdot Pf$, where Pf denotes the map $A \mapsto f(A)$ of type $PX \to PY$. We remark that the monotone map Pf has a left adjoint $Qf : PY \to PX$, $B \mapsto f^{-1}(B)$. Therefore continuity of f is also equivalent to $c \leq Qf \cdot d \cdot Pf$. One quickly verifies the equations

$$\mathcal{C}(f_\#) = Pf \qquad \text{and} \qquad \mathcal{C}(f^\#) = Qf,$$

hence f is continuous if and only if it is consequence preserving with respect to the corresponding consequence relations. Furthermore, a continuous map $f : X \to Y$ between closure spaces (X, c) and (Y, d) is called *initial* whenever $c = Qf \cdot d \cdot Pf$, which corresponds precisely to conservative maps of abstract logics. The connection with topology suggests yet another notion: we call a consequence preserving map $f : X \to Y$ *open* whenever $f^\# \circ \Vdash = \vdash \circ f^\#$. Since $(f^\# \circ \Vdash) \supseteq (\vdash \circ f^\#)$ follows from f being consequence preserving, f is open if and only if $(f^\# \circ \Vdash) \subseteq (\vdash \circ f^\#)$, which in more elementary terms reads as follows: for all $x \in X$ and $B \subseteq Y$ with $B \Vdash f(x)$, there exists $A \subseteq X$ with $A \vdash x$ and $f(A) \subseteq B$.

A coalgebraic view. In this section we will think of an abstract logic $\vdash$ on X as a mapping

$$\alpha := \mathcal{U}(\vdash) : X \to UX,$$

which brings us in the realm of coalgebras. This treatment of logic is motivated by fact that topological spaces can be seen as coalgebras for the filter functor [4], and the subsequent article [7] where closure systems are described as coalgebras for the "contravariant closure system functor". However, our presentation differs slightly from [7] as we consider the up-set functor U (described below) which, moreover, is covariant. As we will see, the latter is not an essential difference since $Uf : UX \to UY$ has an adjoint $Vf : UY \to UX$, for every function $f : X \to Y$.

Here U denotes the up-set functor on Set, where

$$UX = \{\mathcal{A} \subseteq PX \mid \mathcal{A} \text{ is up-closed}\},$$

and a function $f : X \to Y$ is mapped to

$$Uf : UX \to UY, \mathcal{A} \mapsto \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{A}\}.$$

We remark that the monotone map $Uf : UX \to UY$ has a left adjoint $Vf : UY \to UX$ defined by $\mathcal{B} \mapsto \{f^{-1}(B) \mid B \in \mathcal{B}\}$.

Similar to what happened in the previous section, we wish to state now that $\mathcal{U}$ preserves composition, meaning that $\mathcal{U}(s \circ r) = \mathcal{U}(r) * \mathcal{U}(s)$, for monotone relations $r : PX \relbar\joinrel\mapsto Y$ and $PY \relbar\joinrel\mapsto Z$. To do so, we must at first explain the composition $*$ on the right hand side. Here it is useful to note that the functor $U : \mathsf{Set} \to \mathsf{Set}$ comes together with the families of maps $\eta_X : X \to UX$ and $\mu_X : UUX \to UX$ (X is a set) defined by

$$\eta_X(x) = \{A \subseteq X \mid x \in A\} \qquad \text{and} \qquad \mu_X(\mathfrak{A}) = \{A \subseteq X \mid A^{\#} \in \mathfrak{A}\},$$

where $A^{\#} = \{\mathcal{A} \in UX \mid A \in \mathcal{A}\}$. In technical terms, the tripel (U, e, m) is a monad [5].

Given now maps $\sigma : Z \to UY$ and $\rho : Y \to UX$, we put $\rho * \sigma := \mu_X \cdot U\rho \cdot \sigma : Z \to UX$, and with this notation one indeed verifies $\mathcal{U}(s \circ r) = \mathcal{U}(r) * \mathcal{U}(s)$ for all monotone relations $r : PX \relbar\joinrel\mapsto Y$ and $PY \relbar\joinrel\mapsto Z$, as well as $\mathcal{U}(\Delta_X) = \eta_X$ and $\mathcal{U}(r) \leq \mathcal{U}(r')$ for $r \subseteq r'$. Consequently, a map $\alpha : X \to UX$ comes from a consequence relation $\vdash$ on X if and only if

$$\eta_X(x) \subseteq \alpha(x) \qquad \text{and} \qquad \mu_X \cdot U\alpha \cdot \alpha(x) \subseteq \alpha(x),$$

for all $x \in X$. As before, the second inequality is necessarily an equality thanks to the first inequality.

For $f : X \to Y$, we define maps

$$\begin{aligned} f_\diamond : X &\to UY \\ x &\mapsto \{B \subseteq Y \mid f(x) \in B\} \end{aligned} \qquad \text{and} \qquad \begin{aligned} f^\diamond : Y &\to UX \\ y &\mapsto \{A \subseteq X \mid y \in f(A)\}, \end{aligned}$$

and one has

$$\mathcal{U}(f_\#) = f^\diamond \qquad \text{and} \qquad \mathcal{U}(f^\#) = f_\diamond.$$

Furthermore, with $\rho : Y \to UZ$ and $\sigma : Z \to UY$,

$$\rho * f_\diamond = \rho \cdot f \qquad \text{and} \qquad f^\diamond * \sigma = Vf \cdot \sigma$$

Let now $(X, \vdash)$ and $(Y, \Vdash)$ abstract logics with corresponding maps $\alpha : X \to UX$ and $\beta : Y \to UY$. Then f is consequence preserving if and only if

$$\alpha * f^\diamond \leq f^\diamond * \beta,$$

which is equivalent to

$$f_\diamond * \alpha \le \beta * f_\diamond,$$

and this in turn reduces to $Uf \cdot \alpha \le \beta \cdot f$. Moreover, f is conservative if and only if $\alpha = f^\diamond * \beta * f_\diamond$, or, equivalently $\alpha = Vf \cdot \beta \cdot f$. Finally, f is open if and only if $Uf \cdot \alpha = \beta \cdot f$, that is, f is a morphism of coalgebras.

According to the observations above, a coalgebra (X, α) is *induced by an abstract logic* whenever $\eta_X \le \alpha$ and $\alpha * \alpha \le \alpha$. We shall use the same nomenclature as above for a map $f : X \to Y$ between coalgebras (X, α) and (Y, β) in general, that is, f is

- continuous if $Uf \cdot \alpha \le \beta \cdot f$,
- conservative if $\alpha = Vf \cdot \beta \cdot f$ (which is equivalent to $\alpha = f^\diamond * \beta * f_\diamond$),
- progressive if $\beta = f_\diamond * \alpha * f^\diamond$,
- open if $Uf \cdot \alpha = \beta \cdot f$, i.e. f is a morphism of coalgebras.

We use here the new term "progressive" since this concept is somehow dual to conservative. Note that conservative maps are the coalgebra morphisms in the sense of [7], moreover, every conservative map as well as every progressive map is continuous. Also note that every open injection is conservative and every open surjection is progressive.

To finish this subsection, we apply the internal characterisation above to show that the class of coalgebras induced by an abstract logic is a *covariety*, that is, it is closed under homomorphic images, subcoalgebras and sums. To see this, note first that, for a map $f : X \to Y$ and coalgebras (X, α) and (Y, β), α is induced by an abstract logic provided that f is conservative and β is induced by an abstract logic; similarly, β is induced by an abstract logic if f is progressive and α is induced by an abstract logic. This implies at once that the class of coalgebras induced by an abstract logic is closed under homomorphic images and subcoalgebras. To show closedness under the formation of sums, we note that the sum of a family $(X_i, \vdash_i)_{i \in I}$ of abstract logics can be calculated as the disjoint union $X = \coprod_{i \in I} X_i$, equipped with the consequence relation $\vdash$ defined by $A \vdash x$ whenever $(A \cap X_i) \vdash_i x$, where $x \in X_i$ (see also [6]). By definition, every inclusion map $k_i : (X_i, \vdash_i) \hookrightarrow (X, \vdash)$ is open. Hence, $\alpha := \mathcal{U}(\vdash)$ is a coalgebra structure on X making every $k_i : (X_i, \alpha_i) \hookrightarrow (X, \alpha)$ (where $\alpha_i := \mathcal{U}(\vdash_i)$) a coalgebra morphism, and this tells us that the coalgebra (X, α) is the sum of $(X_i, \alpha_i)_{i \in I}$.

3 Conclusion and related work

We have presented several forms how to present a logic, namely via axioms and inference rules, as a consequence relation, via a closure operator and as a coalgebra. This work is not exhaustive; there are further ways of understanding logic in a broad sense. We mention just superficially two more: institution and π-institution. Both of them are based on the principle "truth is invariant under change of notation" [3]. The former is model theoretic while the latter is syntactic. Institutions are the abstract formalization of the process of transforming a first order model into another over a different language by reduction, together with the associated translation of formulae between the two languages which preserves satisfiability. More precisely, assume that $\mathcal{L} \subseteq \mathcal{L}'$ are two first order languages and M is a first-order structure over $\mathcal{L}$. Any first-order formula φ over $\mathcal{L}$ is also a $\mathcal{L}'$-formula, and the "translation" of φ into itself preserves satisfaction in the sense that φ is satisfiable in M if and only if it is satisfiable in the $\mathcal{L}$-reduct of M. The notion of institution was introduced by Goguen and Burstall in [3] to abstract the notion of logical system for which a natural generalisation of this satisfaction condition holds. It also generalises the phenomenon described to the case where the languages are related by a map compatible with the arities of the operation symbols. We should point out that this had not been a research topic for logicians, but the challenges in specification theory of renaming, identifying and adding operation symbols have moved forward its importance; and during last ten years it has been an hot topic of research in both areas: logic and computer science. An institution $I = \langle \mathsf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models \rangle$ consists of an arbitrary category Sign of signatures, a functor SEN from Sign into the category Set of all small sets which defines, for each signature, the set of sentences, a functor MOD from Sign into the opposite of the category CAT of categories, giving, for each signature $\mathcal{L}$, the category $\mathrm{MOD}(\mathcal{L})$ of $\mathcal{L}$-models and, finally, for each signature $\mathcal{L}$, a $\mathcal{L}$-satisfaction relation $\models_{\mathcal{L}}$ between $\mathcal{L}$-models and $\mathcal{L}$-sentences that satisfies the satisfaction condition, which can be summarised in the slogan "truth is invariant under change of notation". There are several examples of institutions. The well known first-order logic and equational logic can be formulated to fit the institution framework.

Fiadeiro and Sernadas [2] presented an alternative formalism: π-institution. The π-institutions provided another manner to deal with deductive systems. It replaces the model-theoretic aspect by the syntactic notion of consequence relation between sets of sentences and individual sentences. However, the multiple languages still is an important ingredient that allows to handle substitutions and, moreover, translations at the language level. A π-institution $I = \langle \mathsf{Sign}, \mathrm{SEN}, (C)_{\Sigma \in |\mathsf{Sign}|} \rangle$ consists of an arbitrary category Sign of signatures, a functor $\mathrm{SEN} : \mathsf{Sign} \to \mathsf{Set}$ (as in the institution case) and, for each signature Σ, a closure operator C_Σ on the set $\mathrm{SEN}(\Sigma)$ of Σ-sentences, such that the following generalized structurality condition holds:

for all $\Sigma, \Sigma' \in |\mathsf{Sign}|$, all $f \in \mathsf{Sign}(\Sigma, \Sigma')$ and all $\Phi \cup \{\varphi\} \in \mathrm{SEN}(\Sigma)$

$$\varphi \in C_\Sigma(\Phi) \text{ implies } \mathrm{SEN}(f)(\varphi) \in C'_\Sigma(\mathrm{SEN}(f)(\Phi)).$$

The algebraic view of a logic, which adds an algebraic structure on the set of formulae to the consequence relation, constitutes an extra challenge to the coalgebraic treatment we have present in this work since a logic in this sense may not be anymore represented by a coalgebra. As already pointed out in [7], bialgebras seem to be the appropriated structures for this case. We are confident that the way we present here can be generalised in this sense in order to captured also the algebraic aspect of logic.

References

1. W.J. Blok and D. Pigozzi. Algebraizable logics. *Memoirs of the American Mathematical Society, American Mathematical Society, Providence*, 77(396):78, 1989.
2. Fiadeiro, J. and Sernadas, A.: *Structuring theories on consequence*, in D. Sannella and A. Tarlecki (eds.), Recent Trends in Data Type Specification, Lecture Notes in Comput. Sci. 332, Springer-Verlag, New York, 1988, pp. 44-72.
3. Goguen, J. A. and Burstall, R. M.: *Institutions: Abstract model theory for specification and programming*, J. Assoc. Comput. Mach. 39(1) (1992), 95-146.
4. H.P. Gumm.: Functors for coalgebras. *Algebra Universalis*, 45(2-3):135–147, 2001. Conference on Lattices and Universal Algebra (Szeged, 1998).
5. J. MacDonald and M. Sobral. Aspects of monads. In *Categorical foundations*, volume 97 of *Encyclopedia Math. Appl.*, pages 213–268. Cambridge Univ. Press, Cambridge, 2004.
6. M. Mahmoudi, C. Schubert and W. Tholen. Universality of coproducts in categories of lax algebras. *Appl. Categ. Struct.*, 14(3):243–249, 2006.
7. A. Palmigiano. Abstract logics as dialgebras. *Electr. Notes Theor. Comput. Sci.*, 65(1), 2002.
8. C. Schubert and G.J. Seal.: Extensions in the theory of lax algebras. *Theory Appl. Categ.*, 21:No. 7, 118–151, 2008.
9. R. Wójcicki. *Theory of logical caculi. Basic theory of consequence operations.* Synthese Library, 199. Dordrecht etc.: Kluwer Academic Publishers., 1988.