Philosophy for philosophers: Metalogic in basic logic

Gabriela Hernández Deciderio
Universidad Nacional Autónoma de México

Summary. Getting metalogic elements into the teaching of formal logic for philosophers, traditionally focused on learning application of a deductive calculus, allows to appreciate logic from a researching perspective, shows the philosophy that came along with logic and encourages students to understand better the logical system studied. In this paper I present an offer for a propositional logic course.

Key words. Metalogic, propositional logic, proof.

Introduction

The education of basic formal logic, aimed at master degree students of philosophy, has been based on texts with mathematical orientation[1] for generations, emphasizing in the procedural aspect of logic. This situation has been changing through the years and there have appeared texts that show, from a basic logic, a logic nearest to a semantic perspective that it does not limit itself to the application of a calculus[2]. Nevertheless, it is possible to give more philosophy to the philosophers in the learning of basic logic through the use of metalogic elements. At least, that is the main motivation and conviction of this work.

Metalogic possesses an eminent philosophical nature that reviews the foundations of logical systems, since its principal desire is to gain assurance that logical systems are reliable. When metalogic is introduced into the basic logic, one hopes to achieve three specific outcomes: 1. Show that philosophical reflection accompanies logic. 2. Reach a better comprehension of the logical system under study 3. Introduce the logic in an investigation perspective, stimulating his development, as much in the purely logical investigation as in his relation with other philosophy areas. The concrete proposal came out in a four-month course of the most basic logic: the propositional logic, for being the logic on which generally starts the education in formal logic.[3] The strategy contemplates five stages, but before detailing them is suitable to justify the selection of the metalogic elements that will be reviewed, and after I give the details of the strategy, and finally I will outline the expected effects from philosophy students.

How much metalogic to introduce?

Locating ourselves in a course of basic logic supposes to think about students who have no previous education in formal logic, vicissitude that hinders the possibility of executing rigorous metalogic proofs; but, for the proposed goals, it is not necessary performing such proofs. The main purpose, ideally, is to arise in the students the desire to improve his knowledge of logic up to construct these proofs. Then, it will be enough with the offering of metalogic expositions, as well as some of its demonstration procedures, without introducing formal definitions, leaning of descriptions, questions and examples that appeal to his intuition. Now then, what metalogic elements to consider? The guideline will be given for what we can identify as the core of the metalogic research: check the balance between the syntactic and semantic expositions of consequence of the system that is been examinig, in this case, of propositional logic. The interest in reaching this core explains why the metalogic takes the logical system checked as his own object of study and thereby evaluate its

[1] Between these classic manuals we find Enderton 2006, Mendelson 1997, Mates 1970, Suppes 1966.
[2] Examples of these manuals are Manzano 2004, GAMUT 2002, Falguera1999, Redmon 1999.
[3] For this proposal, the natural deduction calculus is considered as a model of demonstration, considering it the most helpful for the purpose. However, the proposal could be reconsidered for other calculations as well as other logic systems.

properties. Let's check the meaning of metalogic core with more detail.
As we know, through the propositional semantic, as that of any logical system, it is possible to select, from the set of all formulas of his language, the valid ones[4]. In turn, from the syntactic perspective, we can incorporate into our formal language a deductive calculus in order to obtain a set of derived formulas or theorems. To check the equivalence between both sets we need to get the result of proving the soundness and the completness of the logical system. In addition, it is useful to identify the set of valid formulas because it describes the own logic, its essence or logicality.[5] Additionally, checking out that our system possesses a complete calculus gives us the guarantee of being facing a calculus of general application.

As we can notice, the metalogic elements that must be present in basic logic have to be the indispensable minimums in order to make students realize the meaning and the need to check the core of metalogic, and thanks to it, having a rapprochement to the mathematical tools, those that metalogic uses to execute its demonstrations.

Specific proposal

As it is usual the presentation of a system of propositional logic basically consist of three stages: the first one, to announce the syntactic constitution of the language; the second one, to introduce his semantic interpretation; and the third one, to offer some deductive calculus. This proposal adopts these stages and includes two more. It adds a previous stage or a preparation to the presentation of the formal language, and a final stage in which it offered an invitation to take into account the meaning of checking out the soundness and the completeness of the system observed. As in this proposal we deal exclusively with the propositional logic, it should be possible to mention the additional metalogic quality of this logical system: the decidable. In that form, it is possible to reach the aim of showing the students what constitutes the core of the metalogic. Following this, I analyze the content of every stage[6].

I. Previous Stage

This stage is key for the following development, in which I will detail the meaning of being placed in a metalogic perspective, clarifying his intention and resources. For such a reason, there will be showed basic notions that are guiding the comprehension of the near topics and allowing to reach a vision of the logic not reduced to the application of a deductive calculus. Since our aim is to make accessible the basic core of the metalogic, it is important for the students to approach themselves to the demonstrative procedures of the metalogic, and distinguish them from the demonstration of the inside of the logical system examined; so it is crucial to make them notice that the demonstrative deductive basis, in both levels of demonstration, is in relation with the recursivity, which regularly links itself with the application of mathematical induction. To reach the purpose of this stage the following points will be stressed: a. Metalogic Perspective. b. Intuitive definition of proof. c. Proofs and deduction in logic and mathematics d. Recursively and mathematical induction in proofs of the inside of the system. e. Recursive definitions and demonstrations in metalogic. Now, I will emphasize the central ideas in those points.

[4] We simply speak about valid formulas, since we have restricted ourselves to the propositional logic, otherwise we should speak about the set of the real judgments in all the structures or suitable models.

[5] To fully understand how the set of valid formulas is the essence of logic, we must place ourselves beyond propositional logic. We must place ourselves in logical systems that can be interpreted and which constitute what we call "theory of a structure", in which each interpretation or model may be different, but their common core is the valid formulas that are true in all structure without describing any particular structure, which describes the very logic.

[6] The presented stages exclusively correspond to the presentation of the strategy, it is understood the presence of other elements as the initial location of logic and his history, that use to be ways of starting a course of basic logic for philosophers.

a) *Metalogic Perspective*. It is indispensable that the students warn that in every moment when realizing an intellectual review labor and analysis of the elements of a logic, we make a work of abstraction and, at that moment, we take that logical system as the object of study, which implies locating us from a metalogic point of view. For a closer check of the elements of the logical system we lean on natural language and also on other elements of analysis from mathematical language, particularly from the theory of sets. This utilization of the abstraction must not turn out strange to the philosophy student because it fits with the philosophical examination that, in general, takes certain distance from its objects of study to approach it theoretically. The only difference is that in metalogic we help ourselves through formal language, and not exclusively from the ordinary language.

b) *Intuitive definition of proof.* One of the principal goal of a logical system is its demonstrative capacity, the same that later is evaluated by the metalogic, that means metalogic realizes proofs from a second level. Therefore, it is convenient that students possess an intuitive notion of proof, even before they make a formal one. In the approaching to the proof notion, the main thing is the student understanding that the proof or demonstration of a formula, of the inside of a logical system, takes place in the consecutive linking of series of formulas which we already had (premises)[7], what is possible due to the fact that a deductive calculus gives the rules for its manipulation. In a demonstrative scheme for natural deduction[8], we can also admit the introduction of formulas well formed of our language that operate as supposition, and must be canceled before coming to the derivative formula. In any of the cases, it takes place a finite attainment of lines of formulas, on which the last is the wished formula.

Offering an approximation of the proof notion, before getting to know the logical system, presents a difficulty, since we cannot appeal directly to notions that at the initial moments of a course sound strange for the students, notions like: deductive calculus, premises, or well formed formulas. The fundamental thing is letting the students see that a succession of elements is given in a proof, which we were already possessing since the beginning, and which ones we are manipulating later with the authorization from a bunch of rules created for that purpose. In that case we can overcome the explanatory difficulty by introducing a comparison with a board game, none in particular, it could be any they know from before. We need to guide them to locate the elements presented on a proof in the context of the game, and the meaning of proof in that context is to realize how a play was made correctly.[9] The aim is that the student realizes that there are initial elements and that the rules of the game authorize the movements.[10]

c) *Proofs and deduction in logic and mathematics.* The accomplishment of a proof shows the deductive character of the worked logic; that is, the production of one reasoning in which the conclusion (the formula that we derive or prove) is the fruit of the formulas that we were already possessing, and of its manipulation with the rules of our calculus, this fact shows the deductive nature of propositional logic. The deductive model of the logic is also the demonstrative model of mathematics, hence his nearness and exchange in his demonstrative resources.

d) *Recursively and mathematical induction in proofs of the inside of the system*. The deductive character present in the proofs within the logical systems encourages them to present recursivity, since we start from initial elements on which we can

[7] Instead of premises they could be axioms, everything depends on the deductive calculation in progress.

[8] As noted earlier, we focus on a demonstration scheme for natural deduction, but you can make adjustments to others calculus.

[9] Any game where you use a board is great, if we think of a game in which each player has advanced in three turns and someone requests a proof that the play was performed correctly. The explanation will refer to two elements, the initial data (counters to move forward and starting location) and the rules of the game, which have validated the movements on the board.

[10] For the following notions it is possible to introduce new comparisons of support, for extension reasons, I will omit them of this writing, since the main thing is to offer the justification of the topics selected and to outline the explanatory orientation of his presentation.

apply the rules of calculus and these will produce new formulas on which we can continue applying the rules. The recursivity present in the proof within a system matches up the two stages of a demonstration procedure more powerful in mathematics called mathematical induction, whose stages are: first, known as basic step, where are established the initial objects that mark the beginning; the second, known as the inductive step, where the instructions allow to build new objects from those established in the basis. The order of the procedure is essential, since we start from the initial objects, those where the instructions are applied for the first time and give us new objects on which we can re-apply instructions, and so on, that way the objects will get its order based on the number of times that the rules are applied to obtain them.

In the background of the application of mathematical induction is the order of the natural numbers, which reflects a fundamental affirmation of these numbers known as the principle of mathematical induction which states that: if the first natural number has a property, and given a any natural number, the next one also has the same property, then all natural numbers have that property. This principle gives us the assurance that the order in which objects are generated recursively is the order of natural numbers. Now it is easy to see that in a proof conducted by the application of the deductive calculus of our system, the steps in the application of mathematical induction are presented, in the base step we find the initial formulas: premise and/or assumptions, and in the inductive step, the rules of deduction are defined.

e) *Recursive definitions and demonstrations in metalogic*. As you can recognize the application of mathematical induction and recursions in the proofs of logic systems, there is another procedure where they are introduced again, called definitions by recursion. In the proofs of our calculus the recursion was presented on its finite form, since we obtained the desired formula in a determinated number of lines, by the other hand, the usual goal of the definitions by recursion is to define infinite sets. In the next step, the recursive definition will be used for the presentation of the language of our system, where that reiterated is the implementation of the instructions that build formulas, on the initial objects, then on their product, and so on. This shows that even from a finite basis, we can construct infinite objects.

The application of mathematical induction also it is a useful tool in proofs of metalogic, but the metalogical proofs are from a higher order than those performed into the logical system, because they are oriented to prove properties of our logic system.

Introduction of the propositional language
The intuitive idea of the application of mathematical induction in recursive definitions is recouped to introduce the propositional language, setting as basic step or initial elements the propositional letters, and as inductive step, the introduction of the common logic connectives, those that complex formulas are generated from.

A new metalogic element that is worth to stand out in this stage is the use of metalanguage in the presentation of our propositional language. It is worthy to explain that it is common to present the formation of our formulas using the metalanguage to differentiate between the characteristic elements of formal language and the language that we use to talk about it. It is convenient to talk about the difference between object language and metalanguage, and to consider the possibility of establishing an unlimited hierarchy of metalanguages, if required.

Introduction of semantics for propositional logic
The assignment of truth values for formulas is another element of the propositional logic where it is possible to resort to the recursive definition, in that way, after the presentation of the semantics to the students, they should be able to explain in what sense is recursive a definition and which are the consequences of it.

Introduction of deductive tools for propositional logic

During the review of a deductive propositional calculus, it has to be recouped the recursivity in the application of its rules and the fulfillment of the mathematical induction steps. In this exhibition I made reference only to a natural deduction calculus, due to the consideration that it is more accessible for philosophy students because its nearness with the ordinary procedure of extracting conclusions from a set of premises, but it is not excluded the possibility of considering any other calculus or any other alternative. Nevertheless, it is recommended an exclusive syntactic resource that will make clear the significance of the order for the accomplishment of a derivation and its nearness with the definition of proof that would facilitate the intuitive comprehension of the proofs of soundness-completeness.

Reflections about soundness-completeness and decidability of propositional logic

In the last part of the course, students would already know all the elements of the propositional system, they would have practiced demonstration of theorems and therefore they will be in the propitious moment to close the metalogic approach that was left in the first stage, formulating questions as: Can we know if the system only gives us valid judgments? How can we guarantee it?

The most important thing is that the student notices what we aim to show is throughout deductions, provided by the calculus of our system, we do not make mistakes and that every formula is a genuine consequence of what we had already pre-established. So we are asking for a soundness proof of our system, the guarantee that if we start from true premises, we will not end with false conclusions, therefore supposes a demonstration that the rules of our calculus do not mislead us. To the student who bears in mind the proofs of mathematical induction it will not be difficult accept that by induction proofs, applied to each of our rules, we can verify that they do not mislead us. It is about proving that all the formulas of a deduction fulfill certain property (to be a consequence of the set of formulas constituted by the premises and the not canceled suppositions presented so far in the deduction). For this, it is enough to demonstrate that if up to a certain point of the deduction (let's say, the line n) all the formulas have reach the property, therefore the next formula will too.

The checking of the soundness of system is a metaproof since it forces us to check the entire system. It is valuable to prove the soundness, but it is necessary to check the inverse process: our calculus allows us to obtain the set of all valid judgments, the guiding questions for this part are: How can we have the guarantee that our system gives us the set of valid possible judgments of our language? How can we know if any real formula in our language is obtainable from our deductive device? Can our deductive calculus be applied in general?

These questions lead us to evaluate the completeness of our system, a completeness proof must give us the assurance that if a formula is a consequence obtained from a set of premises of our language, then it must be derivable in our calculus, or expressed in negative, if a formula was not derivable from a certain set of premises of our language, it could not be its consequence. There are several proof procedures, but thinking only in propositional logic, the task is less difficult compared to other logical systems, because propositional logic is predictable, it has an effective method[11] to establish, on each formula, if it is a tautology or not, that method is known as truth tables, which is directly associated with the semantics[12]. So if we have an effective method to recognize the set of tautologies of our language, to prove completeness we must get the procedure to verify that each of the valid formulas or

[11] An effective method is a finite procedure consisting of a series of instructions in which each of everyone make way to the next one depending on certain factors, without leaving any place to the arbitrary choice.

[12] Easily this could lead to resources to obtain the normal form of the formulas of our language.

tautologies can be proved by our system.

By this time, rather than offering metalogical evidence of proof strategy, it is important that students realize that it is possible to obtain the guarantee we aim for, but above all they have to understand what does it means that our system does not mislead us and to understand that our calculus may have general application. It should be noticed that metalogic properties of soundness, completeness and decidability that possesses the propositional logic makes it extremely stable, but that logic has to pay a price for it: poverty of the expressive power of its language. Finally, because we want this proposal to stimulate research, that is why guide questions had been provided, is desirable to conclude the course by motivating the appereance of questions from the reached conclusions, as exposed, as an example, in the later section of this paper.

Advantages of introducing metalogic in the teaching of basic logic

As we advanced, there are basically three advantages: 1.We show that philosophical reflection accompanies logic. 2. We offer a best comprehension of the logical system studied, and 3. Logic is introduced from a research perspective.

1.With the introduction of metalogic elements, the student is able to appreciate that metalogic possesses all the philosophical characteristics of philosophical research that tries to have a critical bearing on his object of study up to express his essence, in this case, his logicity. The recognition of this critical spirit and the search of guarantees from the properties of our system, shows that logic is not exempt of philosophical reflection.

2.The metalogic exposed in this paper demands the student realizing the logical system studied as an object of study, which causes that he acquires a global vision of the system which he is learning and, from this perspective, locates its purposes and characteristics. This way, he will have a better comprehension of the purposes and implications of the system and will avoid keeping fragmented visions, that sometimes are the reason of the student inability recognizing the kindness of logic and seeing it as a mere game of symbols.

3.The evaluating, critical, philosophical spirit stimulates in the metalogic the attempt to prove properties in the systems, especially the coincidence between the set of true formulas and the set of derivable formulas. But the relation also occurs in the other direction, this is, metalogic reflections also reinforce this critical spirit and encourage it to continue formulating philosophical questions inside and beyond logic. From the introduction in metalogic reflections can spontaneously emerge questions like: what does it happen if we add more logical operators to our system? What does it happen if a logical system loses any of his properties? In what cases or what should it happen for losing properties?, there can be alternative logical systems?[13]

We can say that those are questions that have a profound impact just for the logic itself, but there can also be questions that come out from the logic and spread to other areas of philosophical reflection, questions like: can a formal language express all natural languages truths? What kind of entities are propositions? Does truth formal notion include all the senses of the ordinary speech truth concept? To which one do we attribute truth, to the signs or to the referents of signs? To attribute a truth value, must we have knowledge of the entities mentioned or is it an independent process?

[13] The review of the Logic of History can show us the way logicians have had taking ahead this central questioning of metalogic and others near to this one, and that, in different cases, have taken them from the evaluation of a certain logic system to the design of an alternative system, which reveals that metalogic worries are key in the development of the logical investigation.

Metalogic guiding a basic course of logic allows showing a logical tool that is systematic, possesses a few certain properties, and fulfills demonstrative purposes with a certain range. When a philosophy student appreciates the logical tool that he is learning from that dimension allows him to appropriate it with greater awareness. Additionally, if in the appropriation process he achieves to make himself metalogical questions, formulate answers and recognize the strategies to check them rigorously, then he has approached to the logical research. If the questions he made continue questioning the logical elements of the systems he knows, he would be directing himself to the research in Philosophy of Logic, but if instead he wonders about the others dimensions of the language, the existence or the knowledge; then he would be closer to research in other fields of philosophy.

Bibliography

•ENDERTON, Herbert. *A Mathematical Introduction to Logic.* Academic Press, Second Edition, 2001. Spanish translation by J. A. Amor: *Una introducción matemática a la lógica.* México, UNAM-IIFs, 2006.

•FALGUERA López, José L. *Lógica Clásica de Primer Orden: Estrategias de Deducción,Formalizacióny Evaluación Semántica,* Madrid: Trotta, 1999.

•GAMUT, L. T. F. *Logic, language, and meaning: introduction to logic.* 1991 Spanish translation by Cecilia Quirón: *Introducción a la lógica.* Buenos Aires, Eudeba, 2002.

•MANZANO, María y Antonia Huertas. *Lógica para principiantes,* Madrid, Alianza

•MATES, Benson *Elementary Logic,* 1965. Spanish translation: *Lógica matemática elemental,* Tecnos 1970.

•MENDELSON, Elliott, *Introduction to Mathematical Logic,* Wandsworth and Brooks/Cole Advanced Books and Software, Monterrey, California, 1997.

•REDMOND, Walter. *Lógica simbólica para todos.* México, Universidad Veracruzana, 1999.

•SUPPES, Patrick, *Introduction to Logic*. New York: Van Nostrand, 1957. Spanish translation by G. A. Carrasco: *Introducción a la lógica simbólica*. México: Compañía Editorial Continental, SA., 1966.